

# Limit $T$ -subspaces and the central polynomials in $n$ variables of the Grassmann algebra

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## Abstract

Let  $F\langle X \rangle$  be the free unitary associative algebra over a field  $F$  on the set  $X = \{x_1, x_2, \dots\}$ . A vector subspace  $V$  of  $F\langle X \rangle$  is called a  $T$ -subspace (or a  $T$ -space) if  $V$  is closed under all endomorphisms of  $F\langle X \rangle$ . A  $T$ -subspace  $V$  in  $F\langle X \rangle$  is *limit* if every larger  $T$ -subspace  $W \supsetneq V$  is finitely generated (as a  $T$ -subspace) but  $V$  itself is not. Recently Brandão Jr., Koshlukov, Krasilnikov and Silva have proved that over an infinite field  $F$  of characteristic  $p > 2$  the  $T$ -subspace  $C(G)$  of the central polynomials of the infinite dimensional Grassmann algebra  $G$  is a limit  $T$ -subspace. They conjectured that this limit  $T$ -subspace in  $F\langle X \rangle$  is unique, that is, there are no limit  $T$ -subspaces in  $F\langle X \rangle$  other than  $C(G)$ . In the present article we prove that this is not the case. We construct infinitely many limit  $T$ -subspaces  $R_k$  ( $k \geq 1$ ) in the algebra  $F\langle X \rangle$  over an infinite field  $F$  of characteristic  $p > 2$ . For each  $k \geq 1$ , the limit  $T$ -subspace  $R_k$  arises from the central polynomials in  $2k$  variables of the Grassmann algebra  $G$ .

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## 1 Introduction

Let  $F$  be a field,  $X$  a non-empty set and let  $F\langle X \rangle$  be the free unitary associative algebra over  $F$  on the set  $X$ . Recall that a  $T$ -ideal of  $F\langle X \rangle$  is an ideal closed

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under all endomorphisms of  $F\langle X \rangle$ . Similarly, a  $T$ -subspace (or a  $T$ -space) is a vector subspace in  $F\langle X \rangle$  closed under all endomorphisms of  $F\langle X \rangle$ .

Let  $I$  be a  $T$ -ideal in  $F\langle X \rangle$ . A subset  $S \subset I$  generates  $I$  as a  $T$ -ideal if  $I$  is the minimal  $T$ -ideal in  $F\langle X \rangle$  containing  $S$ . A  $T$ -subspace of  $F\langle X \rangle$  generated by  $S$  (as a  $T$ -subspace) is defined in a similar way. It is clear that the  $T$ -ideal ( $T$ -subspace) generated by  $S$  is the ideal (vector subspace) generated by all the polynomials  $f(g_1, \dots, g_m)$ , where  $f = f(x_1, \dots, x_m) \in S$  and  $g_i \in F\langle X \rangle$  for all  $i$ .

Note that if  $I$  is a  $T$ -ideal in  $F\langle X \rangle$  then  $T$ -ideals and  $T$ -subspaces can be defined in the quotient algebra  $F\langle X \rangle/I$  in a natural way. We refer to [9, 10, 12, 18, 20, 25] for the terminology and basic results concerning  $T$ -ideals and algebras with polynomial identities and to [4, 8, 16, 17, 18] for an account of the results concerning  $T$ -subspaces.

From now on we write  $X$  for  $\{x_1, x_2, \dots\}$  and  $X_n$  for  $\{x_1, \dots, x_n\}$ ,  $X_n \subset X$ . If  $F$  is a field of characteristic 0 then every  $T$ -ideal in  $F\langle X \rangle$  is finitely generated (as a  $T$ -ideal); this is a celebrated result of Kemer [19, 20] that solves the Specht problem. Moreover, over such a field  $F$  each  $T$ -subspace in  $F\langle X \rangle$  is finitely generated; this has been proved more recently by Shchigolev [28]. Very recently Belov [7] has proved that, for each Noetherian commutative and associative unitary ring  $K$  and each  $n \in \mathbb{N}$ , each  $T$ -ideal in  $K\langle X_n \rangle$  is finitely generated.

On the other hand, over a field  $F$  of characteristic  $p > 0$  there are  $T$ -ideals in  $F\langle X \rangle$  that are not finitely generated. This has been proved by Belov [5], Grishin [13] and Shchigolev [26] (see also [6, 14, 18]). The construction of such  $T$ -ideals uses the non-finitely generated  $T$ -subspaces in  $F\langle X \rangle$  constructed by Grishin [13] for  $p = 2$  and by Shchigolev [27] for  $p > 2$  (see also [14]). Shchigolev [27] also constructed non-finitely generated  $T$ -subspaces in  $F\langle X_n \rangle$ , where  $n > 1$  and  $F$  is a field of characteristic  $p > 2$ .

A  $T$ -subspace  $V^*$  in  $F\langle X \rangle$  is called *limit* if every larger  $T$ -subspace  $W \supsetneq V^*$  is finitely generated as a  $T$ -subspace but  $V^*$  itself is not. A *limit  $T$ -ideal* is defined in a similar way. It follows easily from Zorn's lemma that if a  $T$ -subspace  $V$  is not finitely generated then it is contained in some limit  $T$ -subspace  $V^*$ . Similarly, each non-finitely generated  $T$ -ideal is contained in a limit  $T$ -ideal. In this sense limit  $T$ -subspaces ( $T$ -ideals) form a "border" between those  $T$ -subspaces ( $T$ -ideals) which are finitely generated and those which are not.

By [5, 13, 26], over a field  $F$  of characteristic  $p > 0$  the algebra  $F\langle X \rangle$  contains non-finitely generated  $T$ -ideals; therefore, it contains at least one limit  $T$ -ideal. No example of a limit  $T$ -ideal is known so far. Even the cardinality of the set of limit  $T$ -ideals in  $F\langle X \rangle$  is unknown; it is possible that, for a given field  $F$  of characteristic  $p > 0$ , there is only one limit  $T$ -ideal. The non-finitely generated  $T$ -ideals constructed in [1] come closer to being limit than any other known non-finitely generated  $T$ -ideal. However, it is unlikely that these  $T$ -ideals are limit.

About limit  $T$ -subspaces in  $F\langle X \rangle$  we know more than about limit  $T$ -ideals. Recently Brandão Jr., Koshlukov, Krasilnikov and Silva [8] have found the first example of a limit  $T$ -subspace in  $F\langle X \rangle$  over an infinite field  $F$  of characteristic  $p > 2$ . To state their result precisely we need some definitions.

For an associative algebra  $A$ , let  $Z(A)$  denote the centre of  $A$ ,

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}.$$

A polynomial  $f(x_1, \dots, x_n)$  is a *central polynomial* for  $A$  if  $f(a_1, \dots, a_n) \in Z(A)$  for all  $a_1, \dots, a_n \in A$ . For a given algebra  $A$ , its central polynomials form a  $T$ -subspace  $C(A)$  in  $F\langle X \rangle$ . However, not every  $T$ -subspace can be obtained as the  $T$ -subspace of the central polynomials of some algebra.

Let  $V$  be the vector space over a field  $F$  of characteristic  $\neq 2$ , with a countable infinite basis  $e_1, e_2, \dots$  and let  $V_s$  denote the subspace of  $V$  spanned by  $e_1, \dots, e_s$  ( $s = 2, 3, \dots$ ). Let  $G$  and  $G_s$  denote the unitary Grassmann algebras of  $V$  and  $V_s$ , respectively. Then as a vector space  $G$  has a basis that consists of 1 and of all monomials  $e_{i_1}e_{i_2}\dots e_{i_k}$ ,  $i_1 < i_2 < \dots < i_k$ ,  $k \geq 1$ . The multiplication in  $G$  is induced by  $e_i e_j = -e_j e_i$  for all  $i$  and  $j$ . The algebra  $G_s$  is the subalgebra of  $G$  generated by  $e_1, \dots, e_s$ , and  $\dim G_s = 2^s$ . We refer to  $G$  and  $G_s$  ( $s = 2, 3, \dots$ ) as to the infinite dimensional Grassmann algebra and the finite dimensional Grassmann algebras, respectively.

The result of [8] concerning a limit  $T$ -subspace is as follows:

**Theorem 1** ([8]). *Let  $F$  be an infinite field of characteristic  $p > 2$  and let  $G$  be the infinite dimensional Grassmann algebra over  $F$ . Then the vector space  $C(G)$  of the central polynomials of the algebra  $G$  is a limit  $T$ -space in  $F\langle X \rangle$ .*

It was conjectured in [8] that a limit  $T$ -subspace in  $F\langle X \rangle$  is unique, that is,  $C(G)$  is the only limit  $T$ -subspace in  $F\langle X \rangle$ . In the present article we show that this is not the case. Our first main result is as follows.

**Theorem 2.** *Over an infinite field  $F$  of characteristic  $p > 2$  the algebra  $F\langle X \rangle$  contains infinitely many limit  $T$ -subspaces.*

Let  $F$  be an infinite field of characteristic  $p > 0$ . In order to prove Theorem 2 and to find infinitely many limit  $T$ -subspaces in  $F\langle X \rangle$  we first find limit  $T$ -subspaces in  $F\langle X_n \rangle$  for  $n = 2k$ ,  $k \geq 1$ . Let  $C_n = C(G) \cap F\langle X_n \rangle$  be the set of the central polynomials in at most  $n$  variables of the unitary Grassmann algebra  $G$ . Our second main result is as follows.

**Theorem 3.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . If  $n = 2k$ ,  $k \geq 1$ , then  $C_n$  is a limit  $T$ -subspace in  $F\langle X_n \rangle$ . If  $n = 2k+1$ ,  $k > 1$ , then  $C_n$  is finitely generated as a  $T$ -subspace in  $F\langle X_n \rangle$ .*

**Remark.** We do not know whether the  $T$ -subspace  $C_3$  is finitely generated.

Define  $[a, b] = ab - ba$ ,  $[a, b, c] = [[a, b], c]$ . For  $k \geq 1$ , let  $T^{(3,k)}$  denote the  $T$ -ideal in  $F\langle X \rangle$  generated by  $[x_1, x_2, x_3]$  and  $[x_1, x_2][x_3, x_4] \dots [x_{2k-1}, x_{2k}]$  and let  $R_k$  denote the  $T$ -subspace in  $F\langle X \rangle$  generated by  $C_{2k}$  and  $T^{(3,k+1)}$ . Theorem 2 follows immediately from our third main result that is as follows.

**Theorem 4.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . For each  $k \geq 1$ ,  $R_k$  is a limit  $T$ -subspace in  $F\langle X \rangle$ . If  $k \neq l$  then  $R_k \neq R_l$ .*

Now we modify the conjecture made in [8].

**Problem 1.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . Is each limit  $T$ -subspace in  $F\langle X \rangle$  equal to either  $C(G)$  or  $R_k$  for some  $k$ ? In other words, are  $C(G)$  and  $R_k$  ( $k \geq 1$ ) the only limit  $T$ -subspaces in  $F\langle X \rangle$ ?*

In the proof of Theorems 3 and 4 we will use the following theorem that has been proved independently by Bekh-Ochir and Rankin [4], by Brandão Jr., Koshlukov, Krasilnikov and Silva [8] and by Grishin [15]. Let

$$q(x_1, x_2) = x_1^{p-1}[x_1, x_2]x_2^{p-1}, \quad q_k(x_1, \dots, x_{2k}) = q(x_1, x_2) \cdots q(x_{2k-1}, x_{2k}).$$

**Theorem 5** ([4], [8], [15]). *Over an infinite field  $F$  of a characteristic  $p > 2$  the vector space  $C(G)$  of the central polynomials of  $G$  is generated (as a  $T$ -space in  $F\langle X \rangle$ ) by the polynomial*

$$x_1[x_2, x_3, x_4]$$

*and the polynomials*

$$x_1^p, \quad x_1^p q_1(x_2, x_3), \quad x_1^p q_2(x_2, x_3, x_4, x_5), \dots, \quad x_1^p q_n(x_2, \dots, x_{2n+1}), \dots$$

In order to prove Theorems 3 and 4 we need some auxiliary results. Define, for each  $l \geq 0$ ,

$$q^{(l)}(x_1, x_2) = x_1^{p^l-1}[x_1, x_2]x_2^{p^l-1},$$

$$q_k^{(l)}(x_1, \dots, x_{2k}) = q^{(l)}(x_1, x_2) \cdots q^{(l)}(x_{2k-1}, x_{2k}).$$

Recall that  $C_n = C(G) \cap F\langle X_n \rangle$ . To prove Theorem 3 we need the following assertions that are also of independent interest.

**Proposition 6.** *If  $n = 2k$ ,  $k > 1$ , then  $C_n$  is generated as a  $T$ -subspace in  $F\langle X_n \rangle$  by the polynomials*

$$x_1[x_2, x_3, x_4], \quad x_1^p, \quad x_1^p q_1(x_2, x_3), \quad \dots, \quad x_1^p q_{k-1}(x_2, \dots, x_{2k-1})$$

*together with the polynomials*

$$\{q_k^{(l)}(x_1, \dots, x_{2k}) \mid l = 1, 2, \dots\}.$$

*If  $n = 2k + 1$ ,  $k > 1$ , then  $C_n$  is generated as a  $T$ -subspace in  $F\langle X_n \rangle$  by the polynomials*

$$x_1[x_2, x_3, x_4], \quad x_1^p, \quad x_1^p q_1(x_2, x_3), \quad \dots, \quad x_1^p q_k(x_2, \dots, x_{2k+1}).$$

Let  $T^{(3)}$  denote the  $T$ -ideal in  $F\langle X \rangle$  generated by  $[x_1, x_2, x_3]$ . Define  $T_n^{(3)} = T^{(3)} \cap F\langle X_n \rangle$ . We deduce Proposition 6 from the following.

**Proposition 7.** *If  $n = 2k$ ,  $k \geq 1$ , then  $C_n/T_n^{(3)}$  is generated as a  $T$ -subspace in  $F\langle X_n \rangle/T_n^{(3)}$  by the polynomials*

$$x_1^p + T_n^{(3)}, \quad x_1^p q_1(x_2, x_3) + T_n^{(3)}, \quad \dots, \quad x_1^p q_{k-1}(x_2, \dots, x_{2k-1}) + T_n^{(3)} \quad (1)$$

*together with the polynomials*

$$\{q_k^{(l)}(x_1, \dots, x_{2k}) + T_n^{(3)} \mid l = 1, 2, \dots\}. \quad (2)$$

*If  $n = 2k + 1$ ,  $k \geq 1$ , then the  $T$ -subspace  $C_n/T_n^{(3)}$  in  $F\langle X_n \rangle/T_n^{(3)}$  is generated by the polynomials*

$$x_1^p + T_n^{(3)}, \quad x_1^p q_1(x_2, x_3) + T_n^{(3)}, \quad \dots, \quad x_1^p q_k(x_2, \dots, x_{2k+1}) + T_n^{(3)}. \quad (3)$$

**Remarks.** 1. For each  $k \geq 1$ , the limit  $T$ -subspace  $R_k$  does not coincide with the  $T$ -subspace  $C(A)$  of all central polynomials of any algebra  $A$ .

Indeed, suppose that  $R_k = C(A)$  for some  $A$ . Let  $T(A)$  be the  $T$ -ideal of all polynomial identities of  $A$ . Then, for each  $f \in C(A)$  and each  $g \in F\langle X \rangle$ , we have  $[f, g] \in T(A)$ . Since  $[x_1, x_2] \in R_k = C(A)$ , we have  $[x_1, x_2, x_3] \in T(A)$ . It follows that  $T^{(3)} \subseteq T(A)$ .

It is well-known that if a  $T$ -ideal  $T$  in the free unitary algebra  $F\langle X \rangle$  over an infinite field  $F$  contains  $T^{(3)}$  then either  $T = T^{(3)}$  or  $T = T^{(3,n)}$  for some  $n$  (see, for instance, [11, Proof of Corollary 7]). Hence, either  $T(A) = T^{(3)}$  or  $T(A) = T^{(3,n)}$  for some  $n$ . Note that  $T^{(3)} = T(G)$  and  $T^{(3,n)} = T(G_{2n-1})$  (see, for example, [11]) so we have either  $T(A) = T(G)$  or  $T(A) = T(G_{2n-1})$  for some  $n$ .

For an associative algebra  $B$ , we have  $f(x_1, \dots, x_r) \in C(B)$  if and only if  $[f(x_1, \dots, x_r), x_{r+1}] \in T(B)$ . It follows that if  $B_1, B_2$  are algebras such that  $T(B_1) = T(B_2)$  then  $C(B_1) = C(B_2)$ . In particular, if  $T(A) = T(G)$  then  $C(A) = C(G)$ , and if  $T(A) = T(G_{2n-1})$  then  $C(A) = C(G_{2n-1})$ .

However,

$$x_1[x_2, x_3] \dots [x_{2k+2}, x_{2k+3}] \in R_k \setminus C(G)$$

so  $R_k \neq C(G)$ . Furthermore, the  $T$ -subspaces  $C(G_s)$  of the central polynomials of the finite dimensional Grassmann algebras  $G_s$  ( $s = 2, 3, \dots$ ) have been described recently by Bekh-Ochir and Rankin [3] and by Koshlukov, Krasilnikov and Silva [21]; these  $T$ -subspaces are finitely generated and do not coincide with  $R_k$ . This contradiction proves that  $R_k \neq C(A)$  for any algebra  $A$ , as claimed.

2. For an associative unitary algebra  $A$ , let  $C_n(A)$  and  $T_n(A)$  denote the set of the central polynomials and the set of the polynomial identities in  $n$  variables  $x_1, \dots, x_n$  of  $A$ , respectively; that is,  $C_n(A) = C(A) \cap F\langle X_n \rangle$  and  $T_n(A) = T(A) \cap F\langle X_n \rangle$ . Then  $C_n(A)$  is a  $T$ -subspace and  $T_n(A)$  is a  $T$ -ideal in  $F\langle X_n \rangle$ .

Note that, by Belov's result [7], the  $T$ -ideal  $T_n(A)$  is finitely generated for each algebra  $A$  over a Noetherian ring and each positive integer  $n$ . On the other hand, there exist unitary algebras  $A$  over an infinite field  $F$  of characteristic  $p > 2$  such that, for some  $n > 1$ , the  $T$ -subspace  $C_n(A)$  of the central polynomials

of  $A$  in  $n$  variables is not finitely generated. Moreover, such an algebra  $A$  can be finite dimensional. Indeed, take  $A = G_s$ , where  $s \geq n$ . It can be checked that  $C(G_s) \cap F\langle X_n \rangle = C_n$  if  $s \geq n$ . By Proposition 9, the  $T$ -subspace  $C_{2k}(G_s)$  in  $F\langle X_{2k} \rangle$  is not finitely generated provided that  $s \geq 2k$ .

However, the following problem remains open.

**Problem 2.** *Does there exist a finite dimensional algebra  $A$  over an infinite field  $F$  of characteristic  $p > 0$  such that the  $T$ -subspace  $C(A)$  of all central polynomials of  $A$  in  $F\langle X \rangle$  is not finitely generated?*

Note that a similar problem for the  $T$ -ideal  $T(A)$  of all polynomial identities of a finite dimensional algebra  $A$  over an infinite field  $F$  of characteristic  $p > 0$  remains open as well; it is one of the most interesting and long-standing open problems in the area.

## 2 Preliminaries

Let  $\langle S \rangle^{TS}$  denote the  $T$ -subspace generated by a set  $S \subseteq F\langle X \rangle$ . Then  $\langle S \rangle^{TS}$  is the span of all polynomials  $f(g_1, \dots, g_n)$ , where  $f \in S$  and  $g_i \in F\langle X \rangle$  for all  $i$ . It is clear that for any polynomials  $f_1, \dots, f_s \in F\langle X \rangle$  we have  $\langle f_1, \dots, f_s \rangle^{TS} = \langle f_1 \rangle^{TS} + \dots + \langle f_s \rangle^{TS}$ .

Recall that a polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  is called a *polynomial identity* in an algebra  $A$  over  $F$  if  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in A$ . For a given algebra  $A$ , its polynomial identities form a  $T$ -ideal  $T(A)$  in  $F\langle X \rangle$  and for every  $T$ -ideal  $I$  in  $F\langle X \rangle$  there is an algebra  $A$  such that  $I = T(A)$ , that is,  $I$  is the ideal of all polynomial identities satisfied in  $A$ . Note that a polynomial  $f = f(x_1, \dots, x_n)$  is central for an algebra  $A$  if and only if  $[f, x_{n+1}]$  is a polynomial identity of  $A$ .

Let  $f = f(x_1, \dots, x_n) \in F\langle X \rangle$ . Then  $f = \sum_{0 \leq i_1, \dots, i_n} f_{i_1 \dots i_n}$ , where each polynomial  $f_{i_1 \dots i_n}$  is multihomogeneous of degree  $i_s$  in  $x_s$  ( $s = 1, \dots, n$ ). We refer to the polynomials  $f_{i_1 \dots i_n}$  as to the *multihomogeneous components* of the polynomial  $f$ . Note that if  $F$  is an infinite field,  $V$  is a  $T$ -ideal in  $F\langle X \rangle$  and  $f \in V$  then  $f_{i_1 \dots i_n} \in V$  for all  $i_1, \dots, i_n$  (see, for instance, [2, 9, 12, 25]). Similarly, if  $V$  is a  $T$ -subspace in  $F\langle X \rangle$  and  $f \in V$  then all the multihomogeneous components  $f_{i_1 \dots i_n}$  of  $f$  belong to  $V$ .

Over an infinite field  $F$  the  $T$ -ideal  $T(G)$  of the polynomial identities of the infinite dimensional unitary Grassmann algebra  $G$  coincides with  $T^{(3)}$ . This was proved by Krakowski and Regev [22] if  $F$  is of characteristic 0 (see also [23]) and by several authors in the general case, see for example [11].

It is well known (see, for example, [22, 23]) that over any field  $F$  we have

$$\begin{aligned} [g_1, g_2][g_1, g_3] + T^{(3)} &= T^{(3)}; \\ [g_1, g_2][g_3, g_4] + T^{(3)} &= -[g_3, g_2][g_1, g_4] + T^{(3)}; \\ [g_1^m, g_2] + T^{(3)} &= mg_1^{m-1}[g_1, g_2] + T^{(3)} \end{aligned} \tag{4}$$

for all  $g_1, g_2, g_3, g_4 \in F\langle X \rangle$ . Also it is well known (see, for instance, [8, 17]) that a basis of the vector space  $F\langle X \rangle/T^{(3)}$  over  $F$  is formed by the elements of the

form

$$x_{i_1}^{m_1} \cdots x_{i_d}^{m_d} [x_{j_1}, x_{j_2}] \cdots [x_{j_{2s-1}}, x_{j_{2s}}] + T^{(3)}, \quad (5)$$

where  $d, s \geq 0$ ,  $i_1 < \dots < i_d$ ,  $j_1 < \dots < j_{2s}$ .

Define  $T_n^{(3)} = T^{(3)} \cap F\langle X_n \rangle$ . We claim that if  $n < 2i$  then

$$T^{(3,i)} \cap F\langle X_n \rangle = T_n^{(3)}. \quad (6)$$

Indeed, a basis of the vector space  $(F\langle X_n \rangle + T^{(3)})/T^{(3)}$  is formed by the elements of the form (5) such that  $1 \leq i_1 < \dots < i_d \leq n$ ,  $1 \leq j_1 < \dots < j_{2s} \leq n$ . In particular, we have  $2s \leq n$ . On the other hand, it can be easily checked that  $T^{(3,i)}/T^{(3)}$  is contained in the linear span of the elements of the form (5) such that  $s \geq i$ . Since  $n < 2i$ , we have

$$((F\langle X_n \rangle + T^{(3)})/T^{(3)}) \cap (T^{(3,i)}/T^{(3)}) = \{0\},$$

that is,  $T^{(3,i)} \cap F\langle X_n \rangle \subseteq T^{(3)}$ . It follows immediately that  $T^{(3,i)} \cap F\langle X_n \rangle \subseteq T_n^{(3)}$ . Since  $T_n^{(3)} \subseteq T^{(3,i)} \cap F\langle X_n \rangle$  for all  $i$ , we have  $T^{(3,i)} \cap F\langle X_n \rangle = T_n^{(3)}$  if  $n < 2i$ , as claimed.

Let  $F$  be a field of characteristic  $p > 2$ . It is well known (see, for example, [24, 4, 8, 16]) that, for each  $g, g_1, \dots, g_n \in F\langle X \rangle$ , we have

$$\begin{aligned} g^p + T^{(3)} &\text{ is central in } F\langle X \rangle/T^{(3)}; \\ (g_1 \cdots g_n)^p + T^{(3)} &= g_1^p \cdots g_n^p + T^{(3)}; \\ (g_1 + \dots + g_n)^p + T^{(3)} &= g_1^p + \dots + g_n^p + T^{(3)}. \end{aligned} \quad (7)$$

Let  $F$  be an infinite field of characteristic  $p > 2$ . Let  $Q^{(k,l)}$  be the  $T$ -subspace in  $F\langle X \rangle$  generated by  $q_k^{(l)}$  ( $l \geq 0$ ),  $Q^{(k,l)} = \langle q_k^{(l)}(x_1, \dots, x_{2k}) \rangle^{TS}$ . Note that the multihomogeneous component of the polynomial

$$\begin{aligned} q_k^{(l)}(1 + x_1, \dots, 1 + x_{2k}) \\ = (1 + x_1)^{p^l - 1} [x_1, x_2] (1 + x_2)^{p^l - 1} \cdots (1 + x_{2k-1})^{p^l - 1} [x_{2k-1}, x_{2k}] (1 + x_{2k})^{p^l - 1} \end{aligned}$$

of degree  $p^{l-1}$  in all the variables  $x_1, \dots, x_{2k}$  is equal to

$$\gamma q_k^{(l-1)}(x_1, \dots, x_{2k}) = \gamma x_1^{p^{l-1}-1} [x_1, x_2] x_2^{p^{l-1}-1} \cdots x_{2k-1}^{p^{l-1}-1} [x_{2k-1}, x_{2k}] x_{2k}^{p^{l-1}-1},$$

where  $\gamma = \binom{p^l - 1}{p^{l-1} - 1}^{2k} \equiv 1 \pmod{p}$ . It follows that  $q_k^{(l-1)} \in Q^{(k,l)}$  for all  $l > 0$  so  $Q^{(k,l-1)} \subseteq Q^{(k,l)}$ . Hence, for each  $l > 0$  we have

$$\sum_{i=0}^l Q^{(k,i)} = Q^{(k,l)}. \quad (8)$$

The following lemma is a reformulation of a result of Grishin and Tsybulya [16, Theorem 1.3, item 1)].

**Lemma 8.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . Let  $k \geq 1$ ,  $a_i \geq 1$  for all  $i = 1, 2, \dots, 2k$  and let*

$$m = x_1^{a_1-1} x_2^{a_2-1} \dots x_{2k}^{a_{2k}-1} [x_1, x_2] \dots [x_{2k-1}, x_{2k}] \in F\langle X \rangle.$$

*Suppose that, for some  $i_0$ ,  $1 \leq i_0 \leq 2k$ , we have  $a_{i_0} = p^l b$ , where  $l \geq 0$  and  $b$  is coprime to  $p$ . Suppose also that, for each  $i$ ,  $1 \leq i \leq 2k$ , we have  $a_i \equiv 0 \pmod{p^l}$ . Then*

$$\langle m \rangle^{TS} + T^{(3)} = Q^{(k,l)} + T^{(3)}.$$

### 3 Proof of Propositions 6 and 7

In the rest of the paper,  $F$  will denote an infinite field of characteristic  $p > 2$ .

#### Proof of Proposition 7

Let  $U$  be the  $T$ -subspace of  $F\langle X_n \rangle$  defined as follows:

- i)  $T_n^{(3)} \subset U$ ;
- ii) the  $T$ -subspace  $U/T_n^{(3)}$  of  $F\langle X_n \rangle/T_n^{(3)}$  is generated by the polynomials (1) and (2) if  $n = 2k$  and by the polynomials (3) if  $n = 2k + 1$ .

To prove the proposition we have to show that  $C_n/T_n^{(3)} = U/T_n^{(3)}$  (equivalently,  $C_n = U$ ). It can be easily seen that  $U/T_n^{(3)} \subseteq C_n/T_n^{(3)}$ . Thus, it remains to prove that  $C_n/T_n^{(3)} \subseteq U/T_n^{(3)}$  (equivalently,  $C_n \subseteq U$ ).

Let  $h$  be an arbitrary element of  $C_n$ . We are going to check that  $h + T_n^{(3)} \in U/T_n^{(3)}$ .

Since  $h \in C(G)$ , it follows from Theorem 5 that

$$h = \sum_j \alpha_j v_j^p + \sum_{i,j} \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) + h',$$

where  $v_j, w_{ij}, f_s^{(ij)} \in F\langle X \rangle$ ,  $\alpha_j, \alpha_{ij} \in F$ ,  $h' \in T^{(3)}$ . Note that  $h \in F\langle X_n \rangle$  so we may assume that  $v_j, w_{ij}, f_s^{(ij)}, h' \in F\langle X_n \rangle$  for all  $i, j, s$ . It follows that

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i,j} \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) + T_n^{(3)}.$$

Recall that  $T^{(3,i)}$  is the  $T$ -ideal in  $F\langle X \rangle$  generated by the polynomials  $[x_1, x_2, x_3]$  and  $[x_1, x_2] \dots [x_{2i-1}, x_{2i}]$ . By (6), we have  $T^{(3,i)} \cap F\langle X_n \rangle = T_n^{(3)}$  for each  $i$  such that  $2i > n$ . Since, for each  $i, j$ ,

$$w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) \in T^{(3,i)},$$



we have

$$\sum_{i > \frac{n}{2}} \sum_j \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) \in T^{(3,i)} \cap F\langle X_n \rangle = T_n^{(3)}.$$

It follows that

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i \leq \frac{n}{2}} \sum_j \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) + T_n^{(3)}.$$

If  $n = 2k + 1$  ( $k \geq 1$ ) then we have

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i=1}^k \sum_j \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)}) + T_n^{(3)}$$

so  $h + T_n^{(3)} \in U/T_n^{(3)}$ , as required.

If  $n = 2k$  ( $k \geq 1$ ) then we have

$$h + T_n^{(3)} = h_1 + h_2 + T_n^{(3)},$$

where

$$h_1 = \sum_j \alpha_j v_j^p + \sum_{i=1}^{k-1} \sum_j \alpha_{ij} w_{ij}^p q_i(f_1^{(ij)}, \dots, f_{2i}^{(ij)})$$

and

$$h_2 = \sum_j \alpha_{kj} w_{kj}^p q_k(f_1^{(kj)}, \dots, f_{2k}^{(kj)}).$$

It is clear that  $h_1 + T_n^{(3)}$  belongs to the  $T$ -subspace generated by the polynomials (1); hence,  $h_1 + T_n^{(3)} \in U/T_n^{(3)}$ . On the other hand, it can be easily seen that  $h_2 + T_n^{(3)}$  is a linear combination of polynomials of the form  $m + T_n^{(3)}$ , where

$$m = x_1^{b_1} \cdots x_{2k}^{b_{2k}} [x_1, x_2] \cdots [x_{2k-1}, x_{2k}].$$

We claim that, for each  $m$  of this form, the polynomial  $m + T_{2k}^{(3)}$  belongs to  $U/T_{2k}^{(3)}$ .

Indeed, by Lemma 8, we have  $\langle m \rangle^{TS} + T^{(3)} = \langle q_k^{(l)} \rangle^{TS} + T^{(3)}$  for some  $l \geq 0$ . Since both  $m$  and  $q_k^{(l)}$  are polynomials in  $x_1, \dots, x_{2k}$ , this equality implies that  $m + T_{2k}^{(3)}$  belongs to the  $T$ -subspace of  $F\langle X_{2k} \rangle / T_{2k}^{(3)}$  that is generated by  $q_k^{(l)} + T_{2k}^{(3)}$  for some  $l \geq 0$ . If  $l \geq 1$  then  $m + T_{2k}^{(3)} \in U/T_{2k}^{(3)}$  because, for  $l \geq 1$ ,  $q_k^{(l)} + T_{2k}^{(3)}$  is a polynomial of the form (2). If  $l = 0$  then  $m + T_{2k}^{(3)}$  belongs to the  $T$ -subspace of  $F\langle X_{2k} \rangle / T_{2k}^{(3)}$  generated by  $q_k^{(1)} + T_{2k}^{(3)}$ . Indeed, in this case  $m + T_{2k}^{(3)}$  belongs to the  $T$ -subspace generated by  $q_k^{(0)} + T_{2k}^{(3)}$  and the latter  $T$ -subspace is contained in the  $T$ -subspace generated by  $q_k^{(1)} + T_{2k}^{(3)}$  because  $q_k^{(0)}$  is equal to the multilinear component of  $q_k^{(1)}(1 + x_1, \dots, 1 + x_{2k})$ . It follows that, again,  $m + T_{2k}^{(3)} \in U/T_{2k}^{(3)}$ . This proves our claim.

It follows that  $h_2 + T_n^{(3)} \in U/T_n^{(3)}$  and, therefore,  $h + T_n^{(3)} \in U/T_n^{(3)}$ , as required.

Thus,  $C_n \subseteq U$  for each  $n$ . This completes the proof of Proposition 7.

## Proof of Proposition 6

It is clear that the polynomial  $x_1[x_2, x_3, x_4]x_5$  generates  $T^{(3)}$  as a  $T$ -subspace in  $F\langle X \rangle$ . Since  $g_1[g_2, g_3, g_4]g_5 = g_1[g_2, g_3, g_4, g_5] + g_1g_5[g_2, g_3, g_4]$  for all  $g_i \in F\langle X \rangle$ , the polynomial  $x_1[x_2, x_3, x_4]$  generates  $T^{(3)}$  as a  $T$ -subspace in  $F\langle X \rangle$  as well. It follows that  $x_1[x_2, x_3, x_4]$  generates  $T_n^{(3)}$  as a  $T$ -subspace in  $F\langle X_n \rangle$  for each  $n \geq 4$ . Proposition 6 follows immediately from Proposition 7 and the observation above.

## 4 Proof of Theorem 3

If  $n = 2k + 1$ ,  $k > 1$ , then Theorem 3 follows immediately from Proposition 6.

Suppose that  $n = 2k$ ,  $k \geq 1$ . Then Theorem 3 is an immediate consequence of the following two propositions.

**Proposition 9.** *For all  $k \geq 1$ ,  $C_{2k}$  is not finitely generated as a  $T$ -subspace in  $F\langle X_{2k} \rangle$ .*

**Proposition 10.** *Let  $k \geq 1$  and let  $W$  be a  $T$ -subspace of  $F\langle X_{2k} \rangle$  such that  $C_{2k} \subsetneq W$ . Then  $W$  is a finitely generated  $T$ -subspace in  $F\langle X_{2k} \rangle$ .*

## Proof of Proposition 9

The proof is based on a result of Grishin and Tsybulya [16, Theorem 3.1].

By Proposition 7,  $C_{2k}$  is generated as a  $T$ -subspace in  $F\langle X_{2k} \rangle$  by  $T_{2k}^{(3)}$  together with the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{k-1}(x_2, \dots, x_{2k-1}) \quad (9)$$

and

$$\{q_k^{(l)}(x_1, \dots, x_{2k}) \mid l = 1, 2, \dots\}.$$

Let  $V_l$  be the  $T$ -subspace of  $F\langle X_{2k} \rangle$  generated by  $T_{2k}^{(3)}$  together with the polynomials (9) and the polynomials  $\{q_k^{(i)}(x_1, \dots, x_{2k}) \mid i \leq l\}$ . Then we have

$$C_{2k} = \bigcup_{l \geq 1} V_l. \quad (10)$$

Also, it is clear that  $V_1 \subseteq V_2 \subseteq \dots$ .

Let  $U^{(k-1)}$  be the  $T$ -subspace in  $F\langle X \rangle$  generated by the polynomials (9). The following proposition is a particular case of [16, Theorem 3.1].

**Proposition 11** ([16]). *For each  $l \geq 1$ ,*

$$(Q^{(k, l+1)} + T^{(3)})/T^{(3)} \not\subseteq (U^{(k-1)} + Q^{(k, l)} + T^{(3, k+1)})/T^{(3)}.$$

**Remark.** The  $T$ -subspaces  $(U^{(k-1)} + T^{(3)})/T^{(3)}$ ,  $(Q^{(k,l)} + T^{(3)})/T^{(3)}$  and  $T^{(3,k+1)}/T^{(3)}$  are denoted in [16] by  $\sum_{i < k} CD_p^{(i)}$ ,  $C_{p^l}^{(k)}$  and  $C^{(k+1)}$ , respectively.

Since the  $T$ -subspace  $Q^{(k,l+1)}$  is generated by the polynomial  $q_k^{(l+1)}$  and  $T^{(3)} \subset T^{(3,k+1)}$ , Proposition 11 immediately implies that

$$q_k^{(l+1)} \notin U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}.$$

Further, since  $T_{2k}^{(3)} \subset T^{(3)} \subset T^{(3,k+1)}$ , we have

$$V_l \subset U^{(k-1)} + \sum_{i \leq l} Q^{(k,i)} + T^{(3,k+1)} = U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}$$

(recall that, by (8),  $\sum_{i \leq l} Q^{(k,i)} = Q^{(k,l)}$ ). It follows that  $q_k^{(l+1)} \notin V_l$  for all  $l \geq 1$ ; on the other hand,  $q_k^{(l+1)} \in V_{l+1}$  by the definition of  $V_{l+1}$ . Hence,

$$V_1 \subsetneq V_2 \subsetneq \dots \quad (11)$$

It follows immediately from (10) and (11) that  $C_{2k}$  is not finitely generated as a  $T$ -subspace in  $F\langle X_{2k} \rangle$ . The proof of Proposition 9 is completed.

## Proof of the Proposition 10

For all integers  $i_1, \dots, i_t$  such that  $1 \leq i_1 < \dots < i_t \leq n$  and all integers  $a_1, \dots, a_n \geq 0$  such that  $a_{i_1}, \dots, a_{i_t} \geq 1$ , define  $\frac{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2} \dots x_{i_t}}$  to be the monomial

$$\frac{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2} \dots x_{i_t}} = x_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \in F\langle X \rangle,$$

where  $b_j = a_j - 1$  if  $j \in \{i_1, i_2, \dots, i_t\}$  and  $b_j = a_j$  otherwise.

**Lemma 12.** Let  $f(x_1, \dots, x_n) \in F\langle X \rangle$  be a multihomogeneous polynomial of the form

$$f = \alpha x_1^{a_1} \dots x_n^{a_n} + \sum_{1 \leq i_1 < \dots < i_{2t} \leq n} \alpha_{(i_1, \dots, i_{2t})} \frac{x_1^{a_1} \dots x_n^{a_n}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] \quad (12)$$

where  $\alpha, \alpha_{(i_1, \dots, i_{2t})} \in F$ . Let  $L = \langle f \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ .

Suppose that  $a_i = 1$  for some  $i$ ,  $1 \leq i \leq n$ . Then either  $L = F\langle X \rangle$  or  $L = \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$  or  $L = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$  for some  $\theta \leq \frac{n-1}{2}$ .

*Proof.* Note that each multihomogeneous polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  can be written, modulo  $T^{(3)}$ , in the form (12). Hence, we can assume without loss of generality (permuting the free generators  $x_1, \dots, x_n$  if necessary) that  $a_1 = 1$ .

Note that if  $\alpha \neq 0$ , then  $f(x_1, 1, \dots, 1) = \alpha x_1 \in L$  so  $L = \langle x_1 \rangle^{TS} = F\langle X \rangle$ . Suppose that  $\alpha = 0$ .

We claim that we may assume without loss of generality that  $f$  is of the form  $f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n)$ , where

$$g = \sum_{\substack{2 \leq i_1 < \dots < i_{2t} \leq n \\ t \geq 1}} \alpha_{(i_1, \dots, i_{2t})} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}]. \quad (13)$$

Indeed, consider a term  $m = \frac{x_1^{a_1} \dots x_n^{a_n}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}]$  in (12). If  $i_1 > 1$  then

$$m = x_1 \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}]. \quad (14)$$

Suppose that  $i_1 = 1$ ; then  $m = m' [x_1, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}]$ , where  $m' = \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_2} \dots x_{i_{2t}}}$ . We have

$$\begin{aligned} m + T^{(3)} &= m' [x_1, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)} \\ &= [m' x_1, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] - x_1 [m', x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)} \\ &= [m' x_1 [x_{i_3}, x_{i_4}] \dots [x_{i_{2t-1}}, x_{i_{2t}}], x_{i_2}] - x_1 [m', x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)}. \end{aligned}$$

Hence,

$$m = -x_1 [m', x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] + h, \quad (15)$$

where  $h \in \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ .

It follows easily from (14) and (15) that there exists a multihomogeneous polynomial  $g_1 = g_1(x_2, \dots, x_n) \in F\langle X \rangle$  such that  $f = x_1 g_1 + h_1$ , where  $h_1 \in \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ . Further, there is a multihomogeneous polynomial  $g$  of the form (13) such that  $g \equiv g_1 \pmod{T^{(3)}}$ ; then  $f = x_1 g + h_2$ , where  $h_2 \in \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ . It follows that  $L = \langle x_1 g(x_2, \dots, x_n) \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ . Thus, we can assume without loss of generality that  $f = x_1 g(x_2, \dots, x_n)$ , where  $g$  is of the form (13), as claimed.

If  $f = 0$  then  $L = \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ . Suppose that  $f \neq 0$ . Let  $\theta = \min \{t \mid \alpha_{(i_1, \dots, i_{2t})} \neq 0\}$ . It is clear that  $2\theta + 1 \leq n$  so  $\theta \leq \frac{n-1}{2}$ . We can assume that  $\alpha_{(2, \dots, 2\theta+1)} \neq 0$ ; then

$$\begin{aligned} f &= x_1 \left( \alpha_{(2, \dots, 2\theta+1)} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_2 \dots x_{2\theta+1}} [x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \right. \\ &\quad \left. + \sum_{\substack{2 \leq i_1 < \dots < i_{2t} \leq n \\ t \geq \theta, i_{2t} > 2\theta+1}} \alpha_{(i_1, \dots, i_{2t})} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] \right). \quad (16) \end{aligned}$$

Let  $f_1(x_1, \dots, x_{2\theta+1}) = f(x_1, x_2, \dots, x_{2\theta+1}, 1, \dots, 1) \in L$ ; then

$$f_1 = \alpha_{(2, \dots, 2\theta+1)} x_1 \frac{x_2^{a_2} \dots x_n^{a_n}}{x_2 \dots x_{2\theta+1}} [x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}].$$

It can be easily seen that the multihomogeneous component of degree 1 in the variables  $x_1, x_2, \dots, x_{2\theta+1}$  of the polynomial  $f_1(x_1, x_2 + 1, \dots, x_{2\theta+1} + 1)$  is equal to

$$\alpha_{(2, \dots, 2\theta+1)} x_1 [x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}].$$

It follows that  $x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \in L$ ; hence,

$$\langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)} \subseteq L.$$

On the other hand, it is clear that the polynomial  $f$  of the form (16) belongs to the  $T$ -subspace of  $F\langle X \rangle$  generated by  $x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}]$ ; it follows that  $\langle f \rangle^{TS} \subseteq \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS}$  and, therefore,

$$L \subseteq \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}.$$

Thus,  $L = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ . The proof of Lemma 12 is completed.  $\square$

**Proposition 13.** *Let  $W$  be a  $T$ -subspace of  $F\langle X_{2k} \rangle$  such that  $C_{2k} \subsetneq W$ . Then  $W = F\langle X_{2k} \rangle$  or  $W$  is generated as a  $T$ -subspace by the polynomials*

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1}),$$

$$x_1[x_2, x_3, x_4], x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}],$$

for some positive integer  $\lambda \leq k-1$ .

*Proof.* It is well-known that over a field  $F$  of characteristic 0 each  $T$ -ideal in  $F\langle X \rangle$  can be generated by its multilinear polynomials. It is easy to check that the same is true for each  $T$ -subspace in  $F\langle X \rangle$ . Over an infinite field  $F$  of characteristic  $p > 0$  each  $T$ -ideal in  $F\langle X \rangle$  can be generated by all its multihomogeneous polynomials  $f(x_1, \dots, x_n)$  such that, for each  $i$ ,  $1 \leq i \leq n$ ,  $\deg_{x_i} f = p^{s_i}$  for some integer  $s_i$  (see, for instance, [2]). Again, the same is true for each  $T$ -subspace in  $F\langle X \rangle$ .

Let  $f(x_1, \dots, x_{2k}) \in W \setminus C_{2k}$  be an arbitrary multihomogeneous polynomial such that, for each  $i$  ( $1 \leq i \leq 2k$ ), we have either  $\deg_{x_i} f = p^{s_i}$  or  $\deg_{x_i} f = 0$ . We may assume that  $\deg_{x_i} f = p^{s_i}$  for  $i = 1, \dots, l$  and  $\deg_{x_i} f = 0$  for  $i = l+1, \dots, 2k$  (that is,  $f = f(x_1, \dots, x_l)$ ). Then we have

$$f + T_{2k}^{(3)} = \alpha m + \sum_{1 \leq i_1 < \dots < i_{2t} \leq l} \alpha_{(i_1, \dots, i_{2t})} \frac{m}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] + T_{2k}^{(3)},$$

where  $\alpha, \alpha_{(i_1, \dots, i_{2t})} \in F$ ,  $m = x_1^{p^{s_1}} \dots x_l^{p^{s_l}}$ .

If  $s_i > 0$  for all  $i = 1, \dots, l$  then it can be easily seen that  $f \in C(G)$  so  $f \in C_{2k}$ , a contradiction with the choice of  $f$ . Thus,  $s_i = 0$  for some  $i$ ,  $1 \leq i \leq l$ . Let  $L_f$  be the  $T$ -subspace of  $F\langle X \rangle$  generated by  $f$ ,  $[x_1, x_2]$  and  $T^{(3)}$ . By Lemma 12, we have either  $L_f = F\langle X \rangle$  or

$$L_f = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$$

for some  $\theta < k$  (since  $f \notin C_{2k}$ , we have  $L_f \neq \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$ ). Note that if  $k = 1$  (that is,  $f = f(x_1, x_2)$ ) then the only possible case is  $L_f = F\langle X \rangle$ .

It is clear that if  $L_f = F\langle X \rangle$  for some  $f \in W \setminus C_{2k}$  then  $x_1 \in W$  so  $W = F\langle X_{2k} \rangle$ . Suppose that  $W \neq F\langle X_{2k} \rangle$ ; then  $k > 1$  and  $L_f \neq F\langle X \rangle$  for all

$f \in W \setminus C_{2k}$ . For each  $f \in W \setminus C_{2k}$  satisfying the conditions of Lemma 12, the  $T$ -subspace  $L_f$  in  $F\langle X \rangle$  can be generated, by Lemma 12, by the polynomials

$$[x_1, x_2], \quad x_1[x_2, x_3x_4] \quad \text{and} \quad x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \quad (17)$$

for some  $\theta = \theta_f < k$ . Since the polynomials (17) belong to  $F\langle X_{2k} \rangle$  (recall that  $k > 1$ ), the  $T$ -subspace in  $F\langle X_{2k} \rangle$  generated by  $f$ ,  $[x_1, x_2]$  and  $T^{(3)}$  is also generated (as a  $T$ -subspace in  $F\langle X_{2k} \rangle$ ) by the polynomials (17). Note that  $[x_1, x_2]$  and  $x_1[x_2, x_3, x_4]$  belong to  $C_{2k}$  so the  $T$ -subspace  $V_f$  in  $F\langle X_{2k} \rangle$  generated by  $f$  and  $C_{2k}$  can be generated by  $C_{2k}$  and  $x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}]$  for some  $\theta = \theta_f < k$ .

Let  $\lambda = \min \{ \theta \mid x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \in W \}$ . Since  $W$  is the sum of the  $T$ -subspaces  $V_f$  for all suitable multihomogeneous polynomials  $f \in W \setminus C_{2k}$  and each  $V_f$  is generated by  $C_{2k}$  and  $x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}]$  for some  $\theta = \theta_f < k$ ,  $W$  can be generated as a  $T$ -subspace in  $F\langle X_{2k} \rangle$  by  $C_{2k}$  and  $x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}]$ . Now it follows easily from Proposition 6 that  $W$  can be generated by the polynomials

$$x_1^p, \quad x_1^p q_1(x_2, x_3), \dots, \quad x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1})$$

together with the polynomials

$$x_1[x_2, x_3, x_4] \quad \text{and} \quad x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}],$$

where we note that  $\lambda < k$ .

This completes the proof of Proposition 13.  $\square$

Proposition 10 follows immediately from Proposition 13. The proof of Theorem 3 is completed.

## 5 Proof of Theorem 4

**Proposition 14.** *For each  $k \geq 1$ ,  $R_k$  is not finitely generated as a  $T$ -subspace in  $F\langle X \rangle$ .*

*Proof.* Recall that  $R_k$  is the  $T$ -subspace in  $F\langle X \rangle$  generated by  $C_{2k}$  and  $T^{(3, k+1)}$ . By Proposition 7,  $C_{2k}$  is generated as a  $T$ -subspace in  $F\langle X_{2k} \rangle$  by  $T_{2k}^{(3)}$  together with the polynomials (9) and the polynomials  $\{q_k^{(l)}(x_1, \dots, x_{2k}) \mid l = 1, 2, \dots\}$ . Since  $T_{2k}^{(3)} \subset T^{(3)} \subset T^{(3, k+1)}$ , we have

$$R_k = U^{(k-1)} + \sum_{l \geq 1} Q^{(k, l)} + T^{(3, k+1)},$$

where  $U^{(k-1)}$  and  $Q^{(k, l)}$  are the  $T$ -subspaces in  $F\langle X \rangle$  generated by the polynomials (9) and by the polynomial  $q_k^{(l)}(x_1, \dots, x_{2k})$ , respectively.

Let  $V_l = U^{(k-1)} + \sum_{i \leq l} Q^{(k,i)} + T^{(3,k+1)}$ . Then

$$R_k = \bigcup_{l \geq 1} V_l \quad (18)$$

and  $V_1 \subseteq V_2 \subseteq \dots$ . Recall that, by (8),  $\sum_{i \leq l} Q^{(k,i)} = Q^{(k,l)}$  so  $V_l = U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}$ . By Proposition 11,  $Q^{(k,l+1)} \not\subseteq V_l$  for all  $l \geq 1$  so

$$V_1 \subsetneq V_2 \subsetneq \dots \quad (19)$$

The result follows immediately from (18) and (19).  $\square$

**Lemma 15.** *Let  $f = f(x_1, \dots, x_n) \in F\langle X \rangle$  be a multihomogeneous polynomial of the form*

$$f = \alpha x_1^{p^{s_1}} \dots x_n^{p^{s_n}} + \sum_{i_1 < \dots < i_{2t}} \alpha_{(i_1, \dots, i_{2t})} \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}], \quad (20)$$

where  $\alpha, \alpha_{(i_1, \dots, i_{2t})} \in F$ ,  $s_i \geq 0$  for all  $i$ . Let  $L = \langle f \rangle^{TS} + R_k$ ,  $k \geq 1$ . Then one of the following holds:

1.  $L = F\langle X \rangle$ ;
2.  $L = R_k$ ;
3.  $L = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$  for some  $\theta$ ,  $1 \leq \theta \leq k$ ;
4.  $L = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k$  for some  $s \geq 1$ .

*Proof.* Note that each multihomogeneous polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  of degree  $p^{s_i}$  in  $x_i$  ( $1 \leq i \leq n$ ) can be written, modulo  $T^{(3)}$ , in the form (20). Hence, we can assume without loss of generality (permuting the free generators  $x_1, \dots, x_n$  if necessary) that  $s_1 \leq s_i$  for all  $i$ . Write  $s = s_1$ .

Suppose that  $s = 0$ . Then, by Lemma 12, we have either

$$\langle f \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)} = F\langle X \rangle$$

or

$$\langle f \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)} = \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$$

or

$$\langle f \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)} = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + \langle [x_1, x_2] \rangle^{TS} + T^{(3)}$$

for some  $\theta$ . Since  $\langle [x_1, x_2] \rangle^{TS} + T^{(3)} \subset R_k$  and  $x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \in R_k$  if  $\theta > k$ , we have either  $L = F\langle X \rangle$  or  $L = R_k$  or

$$L = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$$

for some  $\theta \leq k$ .

Now suppose that  $s > 0$ ; then  $s_i > 0$  for all  $i$ ,  $1 \leq i \leq n$ . It can be easily seen that, by (7),  $x_1^{p^{s_1}} \dots x_n^{p^{s_n}} \in (\langle x_1^p \rangle^{TS} + T^{(3)}) \subset R_k$  and, for all  $t < k$ ,

$$\frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] \in \left( \langle x_1^p q_t(x_2, \dots, x_{2t+1}) \rangle^{TS} + T^{(3)} \right) \subset R_k.$$

Also we have  $\frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] \in T^{(3,k+1)} \subset R_k$  for each  $t > k$ . It follows that we can assume without loss of generality that the polynomial  $f$  is of the form

$$f = \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} \alpha_{(i_1, \dots, i_{2k})} \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}]. \quad (21)$$

Note that if  $n < 2k$  then  $f = 0$  and if  $n = 2k$  then

$$f = \alpha_{(1,2,\dots,2k)} \frac{x_1^{p^{s_1}} \dots x_{2k}^{p^{s_{2k}}}}{x_1 x_2 \dots x_{2k}} [x_1, x_2] \dots [x_{2k-1}, x_{2k}]$$

so, by Lemma 8, we have  $f \in Q^{(k,s)} + T^{(3)}$ , where  $s = s_1 > 0$ . In both cases we have  $f \in R_k$  and  $L = R_k$ .

Suppose that  $n > 2k$ . We claim that we may assume that  $f$  is of the form

$$f(x_1, \dots, x_n) = x_1^{p^s} g(x_2, \dots, x_n), \quad (22)$$

where

$$g = \sum_{2 \leq i_1 < \dots < i_{2k} \leq n} \alpha_{(i_1, \dots, i_{2k})} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}].$$

Indeed, consider a term  $m = \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}]$  in (21). If  $i_1 > 1$  then

$$m = x_1^{p^s} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}]. \quad (23)$$

Suppose that  $i_1 = 1$ . Let  $a_i = p^{s_i}$  for all  $i$ . Then

$$\begin{aligned} m + T^{(3,k+1)} &= x_1^{p^s - 1} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_2} \dots x_{i_{2k}}} [x_1, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)} \\ &= x_{j_1}^{a_{j_1}} \dots x_{j_l}^{a_{j_l}} x_1^{a_1 - 1} \dots x_{i_{2k}}^{a_{i_{2k}} - 1} [x_1, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)} \\ &= x_1^{a_1 - 1} x_{j_1}^{a_{j_1}} \dots x_{j_l}^{a_{j_l}} [x_1, x_{i_2}] x_{i_2}^{a_{i_2} - 1} m' + T^{(3,k+1)}, \end{aligned}$$

where

$$m' = x_{i_3}^{a_{i_3} - 1} [x_{i_3}, x_{i_4}] x_{i_4}^{a_{i_4} - 1} \dots x_{i_{2k-1}}^{a_{i_{2k-1}} - 1} [x_{i_{2k-1}}, x_{i_{2k}}] x_{i_{2k}}^{a_{i_{2k}} - 1},$$



$\{j_1, \dots, j_l\} = \{1, \dots, n\} \setminus \{1, i_2, \dots, i_{2k}\}$ ,  $l = n - 2k > 0$ . Suppose that

$$a_1 = a_{j_1} = a_{j_2} = \dots = a_{j_z} \quad \text{and} \quad a_{j_{z+1}}, a_{j_{z+2}}, \dots, a_{j_l} > a_1.$$

Let

$$u = x_1 x_{j_1} \cdots x_{j_z} x_{j_{z+1}}^{a'_{j_{z+1}}} \cdots x_{j_l}^{a'_{j_l}},$$

where  $a'_i = a_i/p^s$  for all  $i$ . Let

$$h = h(x_1, \dots, x_{2k}) = x_1^{a_1-1} [x_1, x_2] x_2^{a_{i_2}-1} \cdots x_{2k-1}^{a_{i_{2k-1}}-1} [x_{2k-1}, x_{2k}] x_{2k}^{a_{i_{2k}}-1}.$$

By (4),  $h \in C(G)$ ; hence,  $h \in C_{2k} \subset R_k$ . It follows that  $h(u, x_{i_2}, \dots, x_{i_{2k}}) \in R_k$ , that is,

$$u^{p^s-1} [u, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' \in R_k. \quad (24)$$

Since, by (7),  $[v_1^p, v_2] \in T^{(3)} \subset T^{(3,k+1)}$  for all  $v_1, v_2 \in F\langle X \rangle$ , we have

$$\begin{aligned} & u^{p^s-1} [u, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\ &= (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1 x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\ &= (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1, x_{i_2}] x_{j_1} \cdots x_{j_z} x_{i_2}^{a_{i_2}-1} m' \\ &+ (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} x_1 [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\ &= m + x_1^{p^s} x_{j_1}^{p^s-1} \cdots x_{j_z}^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \end{aligned}$$

where the second summand is not present if  $z = 0$  (that is, if  $a_{j_i} > a_1$  for all  $i$ ), in which case  $m \in R_k$ . Since

$$\begin{aligned} & x_1^{p^s} x_{j_1}^{p^s-1} \cdots x_{j_z}^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\ &= x_1^{p^s} \sum_{2 \leq i_1 < \dots < i_{2k}} \beta_{(i_1, \dots, i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)} \end{aligned}$$

for some  $\beta_{(i_1, \dots, i_{2k})} \in F$ , we have

$$m + x_1^{p^s} \sum_{2 \leq i_1 < \dots < i_{2k}} \beta_{(i_1, \dots, i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] \in R_k. \quad (25)$$

It is clear that, using (23) and (25), we can write  $f = f_1 + f_2$ , where

$$f_1 = x_1^{p^s} \left( \sum_{2 \leq i_1 < \dots < i_{2k}} \gamma_{(i_1, \dots, i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] \right)$$

is of the form (22) and  $f_2 \in R_k$ . Then we have  $\langle f \rangle^{TS} + R_k = \langle f_1 \rangle^{TS} + R_k$ . Thus, we can assume (replacing  $f$  with  $f_1$ ) that the polynomial  $f$  is of the form (22), as claimed.

If  $f = 0$  then  $L = R_k$ . Suppose that  $f \neq 0$ . Then we can assume without loss of generality that  $\alpha_{(2,3,\dots,2k+1)} \neq 0$ . It follows that the  $T$ -subspace  $\langle f \rangle^{TS}$  contains the polynomial

$$\begin{aligned} h(x_1, \dots, x_{2k+1}) &= \alpha_{(2,3,\dots,2k+1)}^{-1} f(x_1, \dots, x_{2k+1}, 1, 1, \dots, 1) \\ &= x_1^{p^s} x_2^{p^{s_2}-1} \dots x_{2k+1}^{p^{s_{2k+1}}-1} [x_2, x_3] \dots [x_{2k}, x_{2k+1}]. \end{aligned}$$

Then  $\langle f \rangle^{TS} + R_k$  also contains the homogeneous component of the polynomial  $h(x_1 + 1, \dots, x_{2k+1} + 1)$  of degree  $p^s$  in each variable  $x_i$  ( $i = 1, 2, \dots, 2k + 1$ ), that is equal, modulo  $T^{(3)}$ , to

$$\gamma x_1^{p^s} x_2^{p^{s_2}-1} \dots x_{2k+1}^{p^{s_{2k+1}}-1} [x_2, x_3] \dots [x_{2k}, x_{2k+1}],$$

where  $\gamma = \prod_{i=2}^{2k+1} \binom{p^{s_i}-1}{p^s-1} \equiv 1 \pmod{p}$ . It follows that

$$x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \in \langle f \rangle^{TS} + R_k.$$

On the other hand, for all  $i_1, \dots, i_{2k}$  such that  $2 \leq i_1 < \dots < i_{2k} \leq n$ , we have

$$x_1^{p^s} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} \dots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \dots [x_{i_{2k-1}}, x_{i_{2k}}] \in \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + T^{(3,k+1)}$$

(recall that  $s_i \geq s$  for all  $i$ ) so

$$f \in \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k.$$

Thus,

$$\langle f \rangle^{TS} + R_k = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k,$$

where  $s \geq 1$ . The proof of Lemma 15 is completed.  $\square$

**Proposition 16.** *Let  $W$  be a  $T$ -subspace of  $F\langle X \rangle$  such that  $R_k \subsetneq W$ . Then one of the following holds:*

1.  $W = F\langle X \rangle$ .
2.  $W$  is generated as a  $T$ -subspace by the polynomials

$$\begin{aligned} x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1}), \\ x_1[x_2, x_3, x_4], x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}] \end{aligned}$$

for some  $\lambda \leq k$ .

3.  $W$  is generated as a  $T$ -subspace by the polynomials

$$\begin{aligned} x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{k-1}(x_2, \dots, x_{2k-1}), \\ \{q_k^{(l)}(x_1, \dots, x_{2k}) \mid 1 \leq l \leq \mu - 1\}, x_1^{p^\mu} q_k^{(\mu)}(x_2, \dots, x_{2k+1}), \\ x_1[x_2, x_3, x_4], x_1[x_2, x_3] \dots [x_{2k+2}, x_{2k+3}] \end{aligned}$$

for some  $\mu \geq 1$ .

*Proof.* Let  $f = f(x_1, \dots, x_n)$  be an arbitrary polynomial in  $W \setminus R_k$  satisfying the conditions of Lemma 15, that is, an arbitrary multihomogeneous polynomial such that  $\deg_{x_i} f = p^{s_i}$  for some  $s_i \geq 0$  ( $1 \leq i \leq n$ ). Let  $L_f = \langle f \rangle^{TS} + R_k$ . By Lemma 15, we have either  $L_f = F\langle X \rangle$  or

$$L_f = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$$

for some  $\theta \leq k$  or

$$L_f = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k$$

for some  $s \geq 1$ .

Note that  $W$  is generated as a  $T$ -subspace in  $F\langle X \rangle$  by  $R_k$  together with the polynomials  $f \in W \setminus R_k$  satisfying the conditions of Lemma 15. It follows that  $W = \sum L_f$ , where the sum is taken over all the polynomials  $f \in W \setminus R_k$  satisfying these conditions.

It is clear that if  $L_f = F\langle X \rangle$  for some  $f \in W \setminus R_k$  then  $W = F\langle X \rangle$ . Suppose that  $L_f \neq F\langle X \rangle$  for all  $f \in W \setminus R_k$ . Let, for some  $f \in W \setminus R_k$ , we have  $L_f = \langle x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$ ,  $\theta \leq k$ . Define  $\lambda = \min \{\theta \mid x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \in W\}$ ; note that  $\lambda \leq k$ . We have

$$x_1[x_2, x_3] \dots [x_{2\theta}, x_{2\theta+1}] \in \langle x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS}$$

for all  $\theta \geq \lambda$  and

$$x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \in \langle x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS} + T^{(3)}$$

for all  $s$ . Hence,  $W = \langle x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS} + R_k$ , where  $\lambda \leq k$ . It follows that  $W$  is generated as a  $T$ -subspace by the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1}), \\ x_1[x_2, x_3, x_4], x_1[x_2, x_3] \dots [x_{2\lambda}, x_{2\lambda+1}],$$

$\lambda \leq k$ .

Now suppose that, for all  $f \in W \setminus R_k$  satisfying the conditions of Lemma 15, we have

$$L_f = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k$$

for some  $s = s_f \geq 1$ . Note that if  $s \leq r$  then

$$x_1^{p^r} q_k^{(r)}(x_2, \dots, x_{2k+1}) \in \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + T^{(3)}.$$

Take  $\mu = \min \{s \mid x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \in W\}$ . Then we have  $W = R_k + \langle x_1^{p^\mu} q_k^{(\mu)}(x_2, \dots, x_{2k+1}) \rangle^{TS}$  and it is straightforward to check that  $W$  can be generated as a  $T$ -subspace in  $F\langle X \rangle$  by the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{k-1}(x_2, \dots, x_{2k-1})$$

and the polynomials  $\{q_k^{(l)}(x_1, \dots, x_{2k}) \mid 1 \leq l \leq \mu - 1\}$ ,  $x_1^{p^\mu} q_k^{(\mu)}(x_2, \dots, x_{2k+1})$  together with the polynomials

$$x_1[x_2, x_3, x_4] \quad \text{and} \quad x_1[x_2, x_3] \dots [x_{2k+2}, x_{2k+3}].$$

This completes the proof of Proposition 16.  $\square$

Proposition 16 immediately implies the following corollary.

**Corollary 17.** *Let  $W$  be a  $T$ -subspace of  $F\langle X \rangle$  such that  $R_k \subsetneq W$  ( $k \geq 1$ ). Then  $W$  is a finitely generated  $T$ -subspace in  $F\langle X \rangle$ .*

**Proposition 18.** *If  $k \neq l$  then  $R_k \neq R_l$ .*

*Proof.* Suppose, in order to get a contradiction, that  $R_k = R_l$  for some  $k, l$ ,  $k < l$ . Then we have  $C(G) \subseteq R_l$ .

Indeed, by Theorem 5, the  $T$ -subspace  $C(G)$  is generated by the polynomial  $x_1[x_2, x_3, x_4]$  and the polynomials  $x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_n(x_2, \dots, x_{2n+1}), \dots$ . Clearly,

$$x_1[x_2, x_3, x_4] \in T^{(3)} \subset R_l.$$

Further,

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{l-1}(x_2, \dots, x_{2l-1}) \in R_l$$

by the definition of  $R_l$  and

$$x_1^p q_{k+1}(x_2, \dots, x_{2k+3}), x_1^p q_{k+2}(x_2, \dots, x_{2k+5}), \dots \in T^{(3, k+1)} \subseteq R_k = R_l$$

by the definition of  $T^{(3, k+1)}$ . Since  $k < l$ , we have

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_k(x_2, \dots, x_{2k+1}), x_1^p q_{k+1}(x_2, \dots, x_{2k+3}), \dots \in R_l.$$

Hence, all the generators of the  $T$ -subspace  $C(G)$  belong to  $R_l$  so  $C(G) \subseteq R_l$ , as claimed.

Note that  $T^{(3, k+1)} \subseteq R_l$  and  $T^{(3, k+1)} \not\subseteq C(G)$  so  $C(G) \subsetneq R_l$ . By Theorem 1,  $C(G)$  is a limit  $T$ -subspace so each  $T$ -subspace  $W$  such that  $C(G) \subsetneq W$  is finitely generated. In particular,  $R_l$  is a finitely generated  $T$ -subspace. On the other hand, by Proposition 14, the  $T$ -subspace  $R_l$  is not finitely generated. This contradiction proves that  $R_k \neq R_l$  if  $k \neq l$ , as required.  $\square$

Theorem 4 follows immediately from Proposition 14, Corollary 17 and Proposition 18.

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